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Spectral properties of the Preisach hysteresis model with random input. I. General results

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We derive exact results for the spectral density $S(\omega)$ of the output of the Preisach model, a standard model for complex, nonlocal hysteresis. We obtain general results for uncorrelated input signals with arbitrary input and Preisach densities. It is shown analytically that uncorrelated input signals are transformed into output exhibiting long-time correlations. For the simplest example of uniform input and Preisach distributions we prove that correlations decay asymptotically with a t^{-3} power law corresponding to a logarithmic low frequency divergence of the second derivative of the spectrum $S(\omega)$. A simpler expression for symmetric Preisach models is also obtained, which is discussed in detail in Part II, showing that long-time tails or even $1/f$ noise are general features of this class of models.

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I. INTRODUCTION

Hysteresis is a well-known phenomenon in many branches of science [1]. It refers to situations, where for a given external input parameter or field multiple internal system states are possible. Which one of these states is assumed, depends on previous parameter variations and therefore on the history of the system. Classical examples are magnetic materials, where magnetization and external magnetic field are hysteretically related [2,3]. Sometimes one encounters bistable situations corresponding to only one single hysteresis loop describing the input-output or field vs state relation. We are, however, not interested in these simple systems but in complex hysteretic systems with nonlocal memory resulting in arbitrary many internal states corresponding to a given value of the external field or input. Accordingly, these systems, in addition to a major hysteresis loop, show subloops, sub-sub-loops, etc. as the input is varied. Apart from magnetic materials such behavior is ubiquitously found in all kinds of systems, e.g., shape memory alloys [4,5], piezoelectric materials [6], superconducting systems [7,8], porous materials such as soils [9] or foils [10], consolidated materials [11], and also in economic systems [12,13]. The complex behavior of such systems is often difficult to access from first principles. A phenomenological model, which is very successfully applied to all of the mentioned systems, is the so-called Preisach model. It was introduced in the context of magnetic systems by Weiss and de Freudenreich almost a century ago [14] and became popular through the work of Preisach [15]. The universal properties of this model and its limitations were elaborated in detail in [16,17]. In the mathematical literature the corresponding operator, the so-called Preisach operator, which maps input time series to output time series, also found much interest [18–20]. Despite this there do not exist many rigorous results on the characteristic properties of the output time series generated by this model. Especially from an experimental or practical point of view one is interested, for instance, in correlations within the output time series and the corresponding spectral density, respectively. We provide fully analytic results for these quantities. It is shown that under quite general circumstances uncorrelated input is transformed to output signals with algebraically decaying correlations. Our results will be presented in two parts. In the present paper (Part I) the general method for calculating these quantities is presented. The origin of the long-time tails is demonstrated here explicitly for a simple case of the Preisach operator with asymmetric elementary hysteresis loops. In a second paper (Part II, [21]) we present results for systems with symmetric elementary loops. Since this case is considerably simpler, we are able to provide there a rather complete picture of the mechanisms for the long-time tails.

II. PREISACH MODEL

The Preisach model is defined by the action of the so-called Preisach operator $\mathcal{P}$ as follows:

$$y(t) = \mathcal{P}[x(t)] = \int\int d\alpha d\beta \;\; \mu(\alpha,\beta)\, s_{\alpha\beta}[x(t)], \tag{1}$$

$s_{\alpha\beta}[x(t)] \in \{-1,+1\}$ is the output of a nonideal relay with initial state $s_{\alpha\beta}(t_0)=s_0$ for a given input time series $x(t)$, $t \geq t_0$. It is characterized by a rectangular elementary hysteresis loop as shown in Fig. 1.

The output of such a relay can be written as

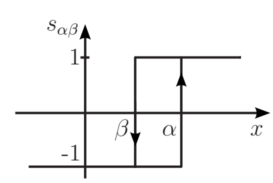

FIG. 1. A rectangular elementary loop characterized by thresholds α and β is shown. The output of the Preisach model is given by a weighted superposition of outputs of such elements.

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$$s_{\alpha\beta}[x(t)] = \begin{cases} +1 & \text{if there exists } t_1 \in [t_0,t] \quad \text{such that } x(t_1) \geq \alpha \text{ and } x(\tau) > \beta \quad \text{for all } \tau \in [t_1,t] \\ -1 & \text{if there exists } t_1 \in [t_0,t] \quad \text{such that } x(t_1) \leq \beta \text{ and } x(\tau) < \alpha \quad \text{for all } \tau \in [t_1,t] \\ s_0 \in \{-1,+1\} & \text{if } \beta < x(\tau) < \alpha \quad \text{for all } \tau \in [t_0,t] \end{cases}. \tag{2}$$

The outputs $s_{\alpha\beta}[x(t)]$ of individual relays with thresholds α and β are weighted with the Preisach density $\mu(\alpha,\beta)$ and summed up to yield the output of the Preisach model according to Eq. (1). Without loss of generality we assume $\alpha > \beta$, or equivalently, $\mu(\alpha,\beta)=0$ for $\alpha \leq \beta$. Equation (1) implies that the output of the Preisach model can be regarded as the superimposed output of infinitely many independent relays connected in parallel [17]. Alternative notions for the Preisach operator are Preisach transducer or Preisach nonlinearity.

Often the input and output variables in Eq. (1) are regarded as variables continuous in time. In the following, however, we consider input sequences in discrete time $t=0,1,2,\ldots$ with elements $\{x(t)\}$. For such time series the action of the Preisach operator is uniquely defined at all integer steps if it is understood that between the discrete time instants the input is supplemented by a continuous time component varying monotonically from $x(t)$ to $x(t+1)$. The exact results presented below are all obtained for $\{x(t)\}$ being a stochastic process consisting of independent identically distributed (i.i.d.) random variables with density $\rho(x)$. The dynamics of the mean output under such assumptions was already considered in the context of viscosity modeling [22].

III. CORRELATION FUNCTIONS

The two-point correlation function is defined as

$$C(t,t+\tau) = \langle y(t)y(t+\tau)\rangle - \langle y(t)\rangle\langle y(t+\tau)\rangle, \tag{3}$$

where $\langle\ldots\rangle$ denotes the average over all input sequences $\{x(t)\}$. For large times t the output correlations become independent of t so that one obtains, as usual, the autocorrelation function $C(\tau)$ of the output signal $y(t)$ for the stationary case as $C(\tau)=\lim_{t\to\infty} C(t,t+\tau)$, which is symmetric in τ; i.e., $C(\tau)=C(-\tau)$ and given by

$$C(\tau) = \langle y(0)y(\tau)\rangle - \langle y\rangle^2. \tag{4}$$

The spectral density $S(\omega)$ is defined as the Fourier transform of $C(\tau)$,

$$S(\omega) = \sum_{\tau=-\infty}^{\infty} C(\tau)e^{i\omega\tau}, \quad -\pi < \omega \leq \pi, \tag{5}$$

which by the Wiener-Khinchin theorem is related to the Fourier transform of the time series $y(t)$ as $S(\omega)=\lim_{N\to\infty}\langle\frac{1}{N}|\sum_{t=1}^{N} y(t)e^{i\omega t}|^2\rangle$. For the calculation of $S(\omega)$, which will be presented in the next section, it is advantageous to make use of the Z transform $\tilde{C}(z)$ of $C(\tau)$ defined for $|z|\geq 1$ as

$$\tilde{C}(z) = \sum_{\tau=0}^{\infty} C(\tau)z^{-\tau}. \tag{6}$$

The latter is related to the spectral density as

$$S(\omega) = 2\,\mathrm{Re}[\tilde{C}(z=e^{i\omega})] - C(0), \tag{7}$$

where the value of the correlation function at time $\tau=0$ is given by $C(0)=\lim_{z\to\infty}\tilde{C}(z)$. Obviously, the time-dependent correlation function $C(\tau)$ can be obtained from $S(\omega)$ by an inverse Fourier transformation, or equivalently, from $\tilde{C}(z)$ by the inverse Z transform.

IV. RESULTS

In this section will first present exact results for the correlation function for arbitrary Preisach densities $\mu(\alpha,\beta)$ and general input densities $\rho(x)$. Explicit analytical results for $\tilde{C}(z)$, the spectrum $S(\omega)$, and the corresponding behavior in the time domain will be given here for one special example, the case of a uniform density in the Preisach plane. In a forthcoming paper our general results will also be applied to a large family of systems, where the Preisach density is concentrated on a line in the Preisach plane.

A. General results

For the calculation of $C(t,t+\tau)$ according to Eq. (3) we need both the time dependence of $\langle y(t)y(t+\tau)\rangle$ and of the mean value $\langle y(t)\rangle$. The latter was obtained already previously in the context of viscosity modeling for magnets and superconductors [17,22]. The evaluation of the autocorrelation function $\langle y(t)y(t+\tau)\rangle$ is more complicated. A method based on diffusion processes on graphs was presented recently in [23] for the Preisach model with symmetric thresholds, which is driven by an Ornstein-Uhlenbeck process. For the latter, however, one has to resume to numerical methods eventually, whereas we are able to calculate the spectral density exactly. This allows us to prove the occurrence of long-time tails in the autocorrelation of the output process.

First note that by using Eq. (1) we can write

$$\langle y(t)y(t+\tau)\rangle = \int\int d\alpha d\beta\, \mu(\alpha,\beta)\int\int d\alpha' d\beta'\, \mu(\alpha',\beta') \times\langle s_{\alpha\beta}[x(t)]s_{\alpha'\beta'}[x(t+\tau)]\rangle. \tag{8}$$

Thus the autocorrelation function of the output $y(t)$ is connected to the cross-correlation function of the output of two elementary relays with thresholds (α,β) and (α',β'), respectively, which are driven by the same stochastic input. The state of the two relays at some discrete time t is described by the state vector $\mathbf{S}(t)=(s_{\alpha\beta}(t),s_{\alpha'\beta'}(t)) \in\{(1,1),(1,-1),(-1,1),(-1,-1)\}\equiv\{\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3,\mathbf{S}_4\}$. It is easy to see that for i.i.d. random input the output of the two relays is described by a four-state Markov process: It is only the current state $\mathbf{S}(t)$ of the two-relay system which determines the probability for the next state $\mathbf{S}(t+1)$. This defines the (4×4)-transition matrix $\mathbf{P}$ with elements $(\mathbf{P})_{ij}=p(\mathbf{S}_j|\mathbf{S}_i)=p(\mathbf{S}(t+1)=\mathbf{S}_j|\mathbf{S}(t)=\mathbf{S}_i)$, the conditional probability for being at time $t+1$ in state $\mathbf{S}_j$, provided the system was in state $\mathbf{S}_i$ at time t. As an example, if the thresholds of the two relays under consideration fulfill the inequalities $\beta'\le\alpha'\le\beta\le\alpha$, the transition probability $(\mathbf{P})_{13}$ is given by $(\mathbf{P})_{13}=p((-1,1)|(1,1))=p(x<\beta\wedge x>\beta')=\int_{\beta'}^{\beta}\rho(x)dx$. This means, in the case that both relays are in the upper state $s=1$ at time t, the one with thresholds α and β flips to $s=-1$ and the other one does not, only if the random input variable $x(t+1)$ falls into the interval $[\beta',\beta]$. In a similar way all elements of the transition matrix $\mathbf{P}$ can be determined. Thus the calculation of the cross-correlation function $\langle s_{\alpha\beta}[x(t)]s_{\alpha'\beta'}[x(t+\tau)]\rangle$ amounts to the determination of a correlation function for a discrete Markov process over four states, which in principle is a straightforward task. One complication comes from the fact that $\mathbf{P}$ depends on the mutual relation between the threshold parameters β', α', β, and α. One has to distinguish six parameter regimes,

$$\begin{aligned}
\text{I:}\quad & \beta'<\alpha'<\beta<\alpha,\\
\text{II:}\quad & \beta'<\beta<\alpha'<\alpha,\\
\text{III:}\quad & \beta'<\beta<\alpha<\alpha',\\
\text{IV:}\quad & \beta<\beta'<\alpha'<\alpha,\\
\text{V:}\quad & \beta<\beta'<\alpha<\alpha',\\
\text{VI:}\quad & \beta<\alpha<\beta'<\alpha'.
\end{aligned}\tag{9}$$

The explicit form of the transition matrices in these regimes $\mathbf{P}_{\mathrm{I}},\ \ldots,\ \mathbf{P}_{\mathrm{VI}}$ is provided in the Appendix. For a given regime with transition matrix $\mathbf{P}$ the evaluation of $\langle s_{\alpha\beta}[x(t)]s_{\alpha'\beta'}[x(t+\tau)]\rangle$ proceeds as follows. The cross-correlation function is defined as

$$\langle s_{\alpha\beta}[x(t)]s_{\alpha'\beta'}[x(t+\tau)]\rangle = \sum_{\substack{s_{\alpha\beta}=\pm1\\ s_{\alpha'\beta'}=\pm1}} s_{\alpha\beta}s_{\alpha'\beta'}p(s_{\alpha\beta},t;s_{\alpha'\beta'},t+\tau),\tag{10}$$

where $p(s_{\alpha\beta},t;s_{\alpha'\beta'},t+\tau)$ is the compound probability to find the relay with thresholds α and β at time t in the state $s_{\alpha\beta}$, and the relay with thresholds α' and β' at time $t+\tau$ in the state $s_{\alpha'\beta'}$. The latter is found from the probability for an arbitrary state sequence of length N of the two-relay system

$$p[\mathbf{S}(1);\mathbf{S}(2);\ldots;\mathbf{S}(N)] = \pi_0\prod_{t=1}^{N}\mathbf{P}(\mathbf{S}(t))\,\eta,\tag{11}$$

where we have introduced the four-component vector of initial probabilities over the states π_0, the vector $\eta=(1,1,1,1)^T$, and the partial transition matrices $\mathbf{P}(\mathbf{S})$, which sum up to the full transition matrix as $\mathbf{P}=\Sigma_{\{\mathbf{S}\}}\mathbf{P}(\mathbf{S})$ and are obtained from the latter by setting to zero all columns which do not describe transitions into the state $\mathbf{S}$. This means $\mathbf{P}(\mathbf{S}_i)$, $i=1,\ldots,4$, coincides with $\mathbf{P}$ in the ith column and all other entries are zero. By summation over all possible events except the ones at times t and $t+\tau$, and assuming without loss of generality $\tau>0$, one obtains $p(\mathbf{S}(t);\mathbf{S}(t+\tau)) =\pi_0\mathbf{P}^{t-1}\mathbf{P}(\mathbf{S}(t))\mathbf{P}^{\tau-1}\mathbf{P}(\mathbf{S}(t+\tau))\,\eta$, and from this expression by further summation $p(s_{\alpha\beta},t;s_{\alpha'\beta'},t+\tau) =\pi_0\mathbf{P}^{t-1}\mathbf{P}(s_{\alpha\beta}(t))\mathbf{P}^{\tau-1}\mathbf{P}(s_{\alpha'\beta'}(t+\tau))\,\eta$. Here we have introduced the matrices $\mathbf{P}(s_{\alpha\beta})=\Sigma_{s_{\alpha'\beta'}=\pm1}\mathbf{P}((s_{\alpha\beta},s_{\alpha'\beta'}))$ and $\mathbf{P}(s_{\alpha'\beta'})=\Sigma_{s_{\alpha\beta}=\pm1}\mathbf{P}((s_{\alpha\beta},s_{\alpha'\beta'}))$. Noting further that $\Sigma_{s_{\alpha\beta}=\pm1}s_{\alpha\beta}\mathbf{P}(s_{\alpha\beta})=\mathbf{P}\cdot\mathbf{I}_1$ and $\Sigma_{s_{\alpha'\beta'}=\pm1}s_{\alpha'\beta'}\mathbf{P}(s_{\alpha'\beta'}) =\mathbf{P}\cdot\mathbf{I}_2$ with the diagonal projection matrices $(\mathbf{I}_1)_{ij} =\delta_{ij}(\delta_{i1}+\delta_{i2}-\delta_{i3}-\delta_{i4})$ and $(\mathbf{I}_2)_{ij}=\delta_{ij}(\delta_{i1}-\delta_{i2}+\delta_{i3}-\delta_{i4})$, one obtains for the cross-correlation function in Eq. (10) the simple expression

$$\langle s_{\alpha\beta}[x(t)]s_{\alpha'\beta'}[x(t+\tau)]\rangle = \pi_0\mathbf{P}^t\mathbf{I}_1\mathbf{P}^\tau\mathbf{I}_2\,\eta,\tag{12}$$

which is valid also for $\tau=0$.

We are especially interested in the stationary case $t\to\infty$. In this limit the vector $\pi_0\mathbf{P}^t$ approaches the stationary probability distribution π^* of the Markov chain, i.e., $\lim_{t\to\infty}\pi_0\mathbf{P}^t=\pi^*$, so that in this case the expression for the stationary cross-correlation function simplifies further to

$$\langle s_{\alpha\beta}[x(0)]s_{\alpha'\beta'}[x(\tau)]\rangle = \pi^*\mathbf{I}_1\mathbf{P}^\tau\mathbf{I}_2\,\eta.\tag{13}$$

We proceed by applying the spectral decomposition of the transition matrix

$$\mathbf{P} = \sum_{r=1}^{4}\lambda_{(r)}\mathbf{u}^{(r)}\otimes\mathbf{v}^{(r)},\tag{14}$$

where $\lambda_{(r)}$ are the eigenvalues of $\mathbf{P}$, and $\mathbf{u}^{(r)}$ and $\mathbf{v}^{(r)}$ are the corresponding right and left eigenvectors, respectively. Inserting the spectral decomposition of $\mathbf{P}$ into Eq. (13) yields, by use of the properties of the dyadic product and the orthogonality relation $\mathbf{v}^{(r)}\cdot\mathbf{u}^{(r')}=\delta_{rr'}$, the result

$$\langle s_{\alpha\beta}[x(0)]s_{\alpha'\beta'}[x(\tau)]\rangle = \sum_{r=1}^{4}\lambda_{(r)}^{\tau}(\pi^*\mathbf{I}_1\mathbf{u}^{(r)})(\mathbf{v}^{(r)}\mathbf{I}_2\,\eta).\tag{15}$$

Note, however, that the spectral decomposition of $\mathbf{P}$ depends on the considered regime [see Eq. (9)]; i.e., we have to perform the decomposition for each of the transition matrices $\mathbf{P}_{\mathrm{I}},\ldots,\mathbf{P}_{\mathrm{VI}}$ given in the Appendix. Here we note that the eigenvalues turn out to be real and non-negative, i.e., $1 =\lambda_{(1)}\ge\lambda_{(2)}\ge\lambda_{(3)}\ge\lambda_{(4)}\ge0$. The explicit general form of all eigenvalues in the six regimes of Eq. (9) is also given in the Appendix. Note that left and right eigenvectors corresponding to $\lambda_{(1)}=1$, which is always an eigenvalue because

$\mathbf{P}$ is a stochastic matrix, are given by $\mathbf{v}^{(1)}=\pi^*$ and $\mathbf{u}^{(1)}=\eta$. This means that $\langle s_{\alpha\beta}[x(0)]s_{\alpha'\beta'}[x(\tau)]\rangle$ decays for large τ to the value $(\pi^*\mathbf{I}_1\eta)(\pi^*\mathbf{I}_2\eta)$, which is just the product $\langle s_{\alpha\beta}\rangle\langle s_{\alpha'\beta'}\rangle$. The explicit expression for the latter is given for completeness $\langle s_{\alpha\beta}\rangle=\frac{F(\alpha,\infty)-F(-\infty,\beta)}{F(\alpha,\infty)+F(-\infty,\beta)}$, but is not needed further because the contribution of $\langle s_{\alpha\beta}\rangle\langle s_{\alpha'\beta'}\rangle$ in the integral Eq. (8) is seen to yield the term $\langle y(t)\rangle^2$, which we want to subtract anyway to obtain the correlation function $C(\tau)$ of Eq. (4). Therefore we can consider the stationary cross-correlation function

$$C_{\alpha\beta,\alpha'\beta'}(\tau)=\langle s_{\alpha\beta}[x(0)]s_{\alpha'\beta'}[x(\tau)]\rangle-\langle s_{\alpha\beta}\rangle\langle s_{\alpha'\beta'}\rangle =\sum_{r=2}^{4}\lambda_{(r)}^{\tau}(\pi^*\mathbf{I}_1\mathbf{u}^{(r)})(\mathbf{v}^{(r)}\mathbf{I}_2\eta), \tag{16}$$

which by integration over the threshold parameter space gives the desired correlation function

$$C(\tau)=\int\int d\alpha d\beta\mu(\alpha,\beta)\int\int d\alpha' d\beta' \mu(\alpha',\beta')C_{\alpha\beta,\alpha'\beta'}(\tau). \tag{17}$$

Despite the appearance in Eq. (16) the explicit spectral decomposition of $\mathbf{P}$ shows that $C_{\alpha\beta,\alpha'\beta'}(\tau)$ decays with only one exponential law corresponding to the eigenvalue $\lambda=F(\beta',\alpha')=\int_{\beta'}^{\alpha'}\rho(x)dx\leq 1$. The contribution from the eigenvalue $F(\beta,\alpha)$ vanishes because of the vanishing of the corresponding coefficients $(\pi^*\mathbf{I}_1\mathbf{u}^{(r)})(\mathbf{v}^{(r)}\mathbf{I}_2\eta)$. The remaining contribution has the simple form

$$C_{\alpha\beta,\alpha'\beta'}(\tau)=C_{\alpha\beta,\alpha'\beta'}(0)F(\beta',\alpha')^{\tau}, \tag{18}$$

where $C_{\alpha\beta,\alpha'\beta'}(0)$ depends on the considered parameter regime as

$$C_{\alpha\beta,\alpha'\beta'}(0)=\frac{4}{[1-F(\beta,\alpha)][1-F(\beta',\alpha')]}\cdot\begin{cases}F(-\infty,\beta')F(\alpha,\infty) & \text{in I,II}\\ F(-\infty,\beta')F(\alpha',\infty) & \text{in III}\\ F(-\infty,\beta)F(\alpha,\infty) & \text{in IV}\\ F(-\infty,\beta)F(\alpha',\infty) & \text{in V,VI}\end{cases}. \tag{19}$$

Thus the correlation function $C(\tau)$ can be regarded as a superposition of infinitely many exponentially decaying contributions in the integral of Eq. (17). Because the relevant eigenvalue $\lambda=F(\beta',\alpha')=\int_{\beta'}^{\alpha'}\rho(x)dx$ can get arbitrarily close to the value $\int_{-\infty}^{\infty}\rho(x)dx=1$ as the parameters β',α' vary in the integral (17), there exists the possibility of a nontrivial, nonexponential decay of $C(\tau)$. That this is indeed the case will be shown below for the simplest case of the Preisach weight function $\mu(\alpha,\beta)$ and the input density $\rho(x)$. But even in this case it is difficult to evaluate the integral in Eq. (17) with the expressions from Eqs. (18) and (19). One better considers the Z transform, Eq. (6), of Eq. (17), i.e.,

$$\tilde{C}(z)=\int\int d\alpha d\beta\mu(\alpha,\beta)\int\int d\alpha' d\beta' \mu(\alpha',\beta')\tilde{C}_{\alpha\beta,\alpha'\beta'}(z), \tag{20}$$

with

$$\tilde{C}_{\alpha\beta,\alpha'\beta'}(z)=C_{\alpha\beta,\alpha'\beta'}(0)\frac{z}{z-F(\beta',\alpha')}. \tag{21}$$

After performing the integration in Eq. (20) exactly, the long-time behavior of $C(\tau)$ can be obtained from the behavior of $\tilde{C}(z)$ for $z\to 1$.

Since in many physical situations the elementary loops can be assumed to be symmetric, i.e., the relay loops of the Preisach model fulfill $\alpha=-\beta$, we give here also the general result for this case. The Preisach density can be expressed with the aid of the Dirac delta distribution $\delta(x)$ as $\mu(\alpha,\beta)=\mu(\alpha)\delta(\alpha+\beta)$. By exploiting the properties of the Dirac delta distribution the expression for the Z transform of the output autocorrelation function, Eq. (20), of the Preisach model simplifies considerably.

$$\begin{aligned}\tilde{C}(z)=&\int_0^{\infty}d\alpha\mu(\alpha)\frac{1}{1-F(-\alpha,\alpha)}\int_{\alpha}^{\infty}d\alpha'\mu(\alpha')\\&\times\frac{F(-\infty,-\alpha')F(\alpha',\infty)}{1-F(-\alpha',\alpha')}\frac{4z}{z-F(-\alpha',\alpha')}\\&+\int_0^{\infty}d\alpha\ \mu(\alpha)\frac{F(-\infty,-\alpha)F(\alpha,\infty)}{1-F(-\alpha,\alpha)}\int_0^{\alpha}d\alpha'\mu(\alpha')\\&\times\frac{1}{1-F(-\alpha',\alpha')}\frac{4z}{z-F(-\alpha',\alpha')}.\end{aligned} \tag{22}$$

Here we used also that only regions III and IV of Eq. (9) contribute to the output. Equation (22) is the basis of a detailed investigation of the symmetric case, which will be presented in a forthcoming paper (Part II, [21]).

B. Special case: Uniform density in the Preisach plane

In the following we treat as an explicit example the case of an equidistributed input density $\rho(x)$ and a uniform Preisach density $\mu(\alpha,\beta)$. To be specific we take $\rho(x)=\frac{1}{2}$ for $-1\leq x\leq 1$ and zero elsewhere. In this case the eigenvalue $F(\beta',\alpha')$ in Eq. (21) and similar terms in Eq. (19) take the simple form $\frac{1}{2}(\alpha'-\beta')$, $\frac{1}{2}(1-\alpha)$, $\frac{1}{2}(\beta+1)$, etc. The Preisach density is assumed to be constant $\mu(\alpha,\beta)=\frac{1}{2}$ inside the triangle $-1\leq\beta\leq\alpha\leq 1$ and zero elsewhere. After changing to new variables $u=1-\alpha$, $v=1-\beta$, $x=1-\alpha'$, $y=1-\beta'$ in the integrals for $\tilde{C}(z)$ in Eq. (20) one obtains

$$\tilde{C}(z) = 2z\Bigg[\int_0^2 du\int_0^{2-u} dv\frac{u}{u+v}\int_{2-v}^2 dx\int_0^{2-x} dy\frac{y}{(x+y)(2z-2+x+y)} + \int_0^2 du\int_0^{2-u} dv\frac{u}{u+v}\int_u^{2-v} dx\int_0^v dy\frac{y}{(x+y)(2z-2+x+y)} + \int_0^2 du\int_0^{2-u} dv\frac{1}{u+v}\int_0^u dx\int_0^v dy\frac{xy}{(x+y)(2z-2+x+y)} + \int_0^2 du\int_0^{2-u} dv\frac{uv}{u+v}\int_u^{2-v} dx\int_v^{2-x} dy\frac{1}{(x+y)(2z-2+x+y)} + \int_0^2 du\int_0^{2-u} dv\frac{v}{u+v}\int_0^u dx\int_v^{2-u} dy\frac{x}{(x+y)(2z-2+x+y)} + \int_0^2 du\int_0^{2-u} dv\frac{v}{u+v}\int_0^u dx\int_{2-u}^{2-x} dy\frac{x}{(x+y)(2z-2+x+y)}\Bigg], \tag{23}$$

where the six terms correspond in its order of appearance to the six parameter regions in Eq. (9). Performing the x and y integration yields after collecting terms the intermediate result

$$\tilde{C}(z) = \frac{z}{6(1-z)}\int_0^2 du\int_0^{2-u} dv\frac{1}{u+v}[10uv(1-z) + 12uv(1-z)\ln(2z) - 8(1-z)^3\ln(2z-2) + u^3\ln u + v^3\ln v - (u+v)^3\ln(u+v) - 3uv(u+v)\ln z - (2z-2+u)^3\ln(2z-2+u) - (2z-2+v)^3\ln(2z-2+v) + (2z-2+u+v)^3\ln(2z-2+u+v)], \tag{24}$$

and after doing the u and v integration and the cancellation of several terms we get the final expression for the Z-transformed Preisach output correlation function

$$\tilde{C}(z) = -\frac{z}{54}(91-303z+186z^2) - \frac{z}{9}(5+19z)(z-1)^2\ln\left(\frac{z-1}{z}\right) - \frac{4}{3}z(z-1)^3\,\mathrm{Li}_2\left(\frac{1}{1-z}\right), \tag{25}$$

where $\mathrm{Li}_2(z)$ is the Euler dilogarithm, which can be represented as $\mathrm{Li}_2(z)=\sum_{n=1}^{\infty}z^n/n^2=-\int_0^z dt\frac{1}{t}\ln(1-t)$. For the calculation of the spectral density we also need $C(t=0)$, which is obtained as $C(t=0)=\lim_{z\to\infty}\tilde{C}(z)=\frac{5}{18}$. The behavior of the corresponding correlation function in the time domain $C(t)$ can, in principle, be obtained by applying the inverse Z transformation to Eq. (25). One is, however, mainly interested in the long-time behavior of $C(t)$. This is most easily obtained by applying a version of Karamata's Tauberian theorem, which connects the behavior of $C(t)$ for large t with that of $\tilde{C}(z)$ near $z=1$. To be precise, we apply the Tauberian theorem for power series as stated in [24]. It says that for $q_n \geq 0$ under the assumption that the power series $Q(s) = \sum_{n=0}^{\infty}q_n s^n$ is convergent for $0\leq s<1$, the following relations

$$Q(s) \sim \frac{1}{(1-s)^r}L\left(\frac{1}{1-s}\right) \quad \text{for } s\to 1- \tag{26}$$

and

$$\sum_{n=0}^{t-1} q_n \sim \frac{1}{\Gamma(r+1)}t^r L(t) \quad \text{for } t\to\infty \tag{27}$$

imply each other. Here $\Gamma(x)$ is the gamma function $0\leq r<\infty$, and $L(x)$ is a slowly varying function at infinity, i.e., $\frac{L(\lambda x)}{L(x)}\to 1$ for $x\to\infty$ and every fixed λ.

To obtain the asymptotic behavior of $\tilde{C}(z)$ near $z=1$ we apply the transformation (see, e.g., [25])

$$\mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(-z) - \mathrm{Li}_2(z) \tag{28}$$

to $\mathrm{Li}_2(\frac{1}{1-z})$ in Eq. (25). Expanding the result around $z=1$ gives

$$\tilde{C}(z) \sim \frac{13}{27} - \frac{43}{54}(z-1) - \left[\frac{85}{18} + \frac{8}{3}\ln(z-1)\right](z-1)^2 + O[(z-1)^3], \tag{29}$$

which shows that the second derivative $\tilde{C}^{(2)}(z)$ of $\tilde{C}(z)$ is diverging as z approaches the value $z=1$. Actually, expanding $\tilde{C}^{(2)}(z)$ around $z=1$ yields

$$\tilde{C}^{(2)}(z) \sim -\frac{16}{3}\ln(z-1) - \frac{157}{9} + O[(z-1)]. \tag{30}$$

Note that $z^2\tilde{C}^{(2)}(z)$ is the Z transform of $t(t+1)C(t)$. By identifying $z^2\tilde{C}^{(2)}(z)|_{z=1/s}$ with $Q(s)$ of Eq. (26) one sees that with Eq. (30) the latter behaves asymptotically for $s\to 1-$ as $Q(s)\sim\frac{16}{3}\ln(\frac{1}{1-s})$ so that according to the above Tauberian theorem with $r=0$ and the slowly varying function $L(x) = \ln(x)$, one obtains from Eq. (27) $\sum_{n=0}^{t-1}n(n+1)C(n)\sim\frac{16}{3}\ln(t)$. The latter relation gives immediately the exact long-time behavior of the output correlation function as

$$C(t) \sim \frac{16}{3}t^{-3}. \tag{31}$$

This remarkable result can also be deduced by an exact inversion of the logarithmic term in Eq. (25). It means that the Preisach transducer, Eq. (1), turns uncorrelated input into

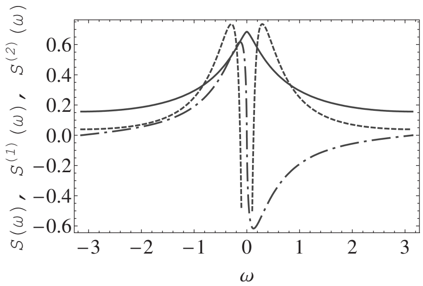

FIG. 2. The spectrum $S(\omega)$ as given by Eq. (7) with Eq. (25) is shown (full line). Its first derivative $S^{(1)}(\omega)$ (dot-dashed) crosses the origin with infinite slope as given by the logarithmic divergence of the second derivative $S^{(2)}(\omega)$ (dashed).

long-time correlated output characterized by a power law decay. In experimental situations instead of the autocorrelation function one often considers the power spectral density $S(\omega)$ of the output time series $y(t)$. The exact form is obtained simply by inserting the result of Eq. (25) into Eq. (7). The result is graphically displayed in Fig. 2.

The behavior of the output-autocorrelation function $C(t)$ obtained by a numerical inverse Fourier transform of the analytic expression Eq. (7) with Eq. (25) is shown in Fig. 3 together with its asymptotic form, Eq. (31).

This long-time behavior is reflected in the small frequency regime of the spectrum. The asymptotic behavior of $S(\omega)$ for $\omega \to 0$ is found as

$$S(\omega) \sim \frac{37}{54} + \frac{16}{3}\omega^2 \ln|\omega| + \frac{553}{54}\omega^2 + O(\omega^3), \qquad (32)$$

which shows that in accordance with the Z transform, Eq. (30), the second derivative $S^{(2)}(\omega)$ of the spectrum diverges logarithmically for $\omega \to 0$. We should mention that this divergence and the corresponding t^{-3} decay of the autocorrelation function can be observed also in direct numerical simulations of the Preisach model [26].

V. CONCLUSION AND DISCUSSION

In this paper we derived general exact expressions for the spectral density of the output of the Preisach model with uncorrelated input. By an explicit calculation we obtained for the special case of a uniform Preisach density that the autocorrelation function $C(t)$ decays with a power law of the

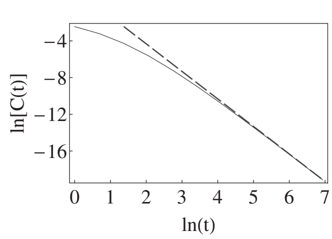

FIG. 3. The output correlation function $C(t)$ obtained from a numerical Fourier inversion of $S(\omega)$ is shown in a double logarithmic plot (full line). For comparison the asymptotic long-time tail according to Eq. (31) is plotted (dashed line).

form $t^{-\eta}$, with $\eta=3$. This means that the Preisach transducer transforms uncorrelated time series into output with infinitely long correlations. Formally this is due to the infinite-dimensional character of the Preisach model, i.e., the internal memory corresponds to infinitely many degrees of freedom. With our results we found a new mechanism for the generation of long-time tails. In the frequency domain it is reflected in a logarithmic divergence of the second derivative of the spectral density. The numerical value of the exponent η will be shown to depend, e.g., on the tails of the probability density of the input. This result is found numerically [26] and can be shown analytically for Preisach models with symmetric elementary hysteresis loops [21]. In the frequency domain this may even result in the appearance of $1/f$ noise. Due to the general character of the Preisach model we expect the same phenomena to be observable also in experimental output time series of complex hysteretic systems with multiple branches.

APPENDIX

We provide for completeness the explicit form of the transition matrices **P** and their eigenvalues in the six parameter regimes of Eq. (9),

$$\mathbf{P}_{\mathrm{I}} = \begin{pmatrix} F(\beta,\infty) & 0 & F(\beta',\beta) & F(-\infty,\beta') \\ F(\beta,\infty) & 0 & F(\alpha',\beta) & F(-\infty,\alpha') \\ F(\alpha,\infty) & 0 & F(\beta',\alpha) & F(-\infty,\beta') \\ F(\alpha,\infty) & 0 & F(\alpha',\alpha) & F(-\infty,\alpha') \end{pmatrix},$$

$$\mathbf{P}_{\mathrm{II}} = \begin{pmatrix} F(\beta,\infty) & 0 & F(\beta',\beta) & F(-\infty,\beta') \\ F(\alpha',\infty) & F(\beta,\alpha') & 0 & F(-\infty,\beta) \\ F(\alpha,\infty) & 0 & F(\beta',\alpha) & F(-\infty,\beta') \\ F(\alpha,\infty) & 0 & F(\alpha',\alpha) & F(-\infty,\alpha') \end{pmatrix},$$

$$\mathbf{P}_{\mathrm{III}} = \begin{pmatrix} F(\beta,\infty) & 0 & F(\beta',\beta) & F(-\infty,\beta') \\ F(\alpha',\infty) & F(\beta,\alpha') & 0 & F(-\infty,\beta) \\ F(\alpha,\infty) & 0 & F(\beta',\alpha) & F(-\infty,\beta') \\ F(\alpha',\infty) & F(\alpha,\alpha') & 0 & F(-\infty,\alpha) \end{pmatrix},$$

$$\mathbf{P}_{\mathrm{IV}} = \begin{pmatrix} F(\beta',\infty) & F(\beta,\beta') & 0 & F(-\infty,\beta) \\ F(\alpha',\infty) & F(\beta,\alpha') & 0 & F(-\infty,\beta) \\ F(\alpha,\infty) & 0 & F(\beta',\alpha) & F(-\infty,\beta') \\ F(\alpha,\infty) & 0 & F(\alpha',\alpha) & F(-\infty,\alpha') \end{pmatrix},$$

$$\mathbf{P}_{\mathrm{V}} = \begin{pmatrix} F(\beta',\infty) & F(\beta,\beta') & 0 & F(-\infty,\beta) \\ F(\alpha',\infty) & F(\beta,\alpha') & 0 & F(-\infty,\beta) \\ F(\alpha,\infty) & 0 & F(\beta',\alpha) & F(-\infty,\beta') \\ F(\alpha',\infty) & F(\alpha,\alpha') & 0 & F(-\infty,\alpha) \end{pmatrix},$$

$$\mathbf{P}_{\mathrm{VI}} = \begin{pmatrix} F(\beta',\infty) & F(\beta,\beta') & 0 & F(-\infty,\beta) \\ F(\alpha',\infty) & F(\beta,\alpha') & 0 & F(-\infty,\beta) \\ F(\beta',\infty) & F(\alpha,\beta') & 0 & F(-\infty,\alpha) \\ F(\alpha',\infty) & F(\alpha,\alpha') & 0 & F(-\infty,\alpha) \end{pmatrix}, \tag{A1}$$

where we have used the abbreviation $F(a,b)=\int_a^b \rho(x)dx$. The general form of these matrices can be obtained as follows. Introducing the characteristic (2×2) matrix of an elementary relay as

$$\mathbf{P}_{\alpha\beta}(x) = \begin{pmatrix} \theta(x-\beta) & \theta(\beta-x) \\ \theta(x-\alpha) & \theta(\alpha-x) \end{pmatrix}, \tag{A2}$$

where $\theta(x)$ is the Heaviside step function, one can express the Markov transition matrix describing the switching properties of a single elementary relay with thresholds α and β driven by uncorrelated random input with density $\rho(x)$ as $\langle \mathbf{P}_{\alpha\beta}(x)\rangle$, where $\langle \ldots \rangle$ denotes the average $\int \ldots \rho(x)dx$. Similarly, under the same input conditions the (4×4)-transition matrix $\mathbf{P}$ describing the simultaneous switching of two parallel elementary relays with thresholds α,β and α',β', respectively, can be written simply as

$$\mathbf{P} = \langle \mathbf{P}_{\alpha\beta}(x) \otimes \mathbf{P}_{\alpha'\beta'}(x)\rangle, \tag{A3}$$

i.e., as expectation of the Kronecker product (see, e.g., [27]) of the two characteristic matrices $\mathbf{P}_{\alpha\beta}(x)$ and $\mathbf{P}_{\alpha'\beta'}(x)$. This is the formal consequence of the fact that in the Preisach model all elementary relays are independent but receive the same input in parallel. The generalization to more than two relays is obviously given by $\langle \mathbf{P}_{\alpha\beta}(x)\otimes \mathbf{P}_{\alpha'\beta'}(x)\otimes \mathbf{P}_{\alpha''\beta''}(x)\rangle$, etc., but this is not needed for the calculation of two-point correlation functions such as Eq. (8). The splitting of $\mathbf{P}$ of Eq. (A3) into the cases $\mathbf{P}_{\mathrm{I}},\ldots,\mathbf{P}_{\mathrm{VI}}$ of Eq. (A1) arises from the different values obtained for the matrix elements $\mathbf{P}_{ij}$ in dependence on the mutual order relations of the thresholds α, β, α', and β'. For instance, $\mathbf{P}_{11}=\langle\theta(x-\beta)\theta(x-\beta')\rangle =\langle\theta(x-\beta)\rangle=F(\beta,\infty)$ for $\beta'<\beta$, but $\mathbf{P}_{11}=\langle\theta(x-\beta')\rangle =F(\beta',\infty)$ for $\beta'>\beta$.

The eigenvalues and (left and right) eigenvectors of these matrices can also be determined explicitly. We list only the eigenvalues $\lambda_{(r)}$, $r=1,2,3,4$ for each of the six regimes of Eq. (9) as follows:

$$\begin{aligned} &\mathrm{I:} && 1, F(\beta,\alpha), F(\beta',\alpha'), 0;\\ &\mathrm{II:} && 1, F(\beta,\alpha), F(\beta,\alpha'), F(\beta',\alpha');\\ &\mathrm{III:} && 1, F(\beta,\alpha), F(\beta,\alpha), F(\beta',\alpha');\\ &\mathrm{IV:} && 1, F(\beta,\alpha), F(\beta',\alpha'), F(\beta',\alpha');\\ &\mathrm{V:} && 1, F(\beta',\alpha'), F(\beta',\alpha), F(\beta,\alpha);\\ &\mathrm{VI:} && 1, F(\beta',\alpha'), F(\beta,\alpha), 0. \end{aligned} \tag{A4}$$

For the eigenvalues one can also find a general expression. The eigenvalues of $\mathbf{P}_{\alpha\beta}(x)$, Eq. (A2), are found as $\lambda_{(1)}(x)=1$ and $\lambda_{(2)}(x)=\chi_{\beta\alpha}(x)$, where $\chi_{\beta\alpha}(x)$ is the characteristic function of the interval $[\beta,\alpha]$, i.e., $\chi_{\beta\alpha}(x) =\theta(x-\beta)-\theta(\alpha-x)$ (note that $\beta<\alpha$). According to the rules for Kronecker products of matrices the eigenvalues of $\mathbf{P}_{\alpha\beta}(x)\otimes\mathbf{P}_{\alpha'\beta'}(x)$ are the products of the eigenvalues of the factors [27] and therefore are given by $\{\lambda_{(r)}(x), r=1,2,3,4\} =\{1,\chi_{\beta\alpha}(x),\chi_{\beta'\alpha'}(x),\chi_{\beta\alpha}(x)\chi_{\beta'\alpha'}(x)\}$. One finds that the eigenvalues of $\mathbf{P}=\langle\mathbf{P}_{\alpha\beta}(x)\otimes\mathbf{P}_{\alpha'\beta'}(x)\rangle$ are simply the expectation values of the latter, i.e.,

$$\{\lambda_{(r)}, r=1,2,3,4\}=\{1,\langle\chi_{\beta\alpha}(x)\rangle,\langle\chi_{\beta'\alpha'}(x)\rangle,\langle\chi_{\beta\alpha}(x)\chi_{\beta'\alpha'}(x)\rangle\}. \tag{A5}$$

This simple result can be verified by a direct comparison of Eqs. (A4) and (A5) in the six different parameter regimes. It can also be obtained by a direct computation exploiting the Kronecker product structure $\mathbf{P}_{\alpha\beta}(x)\otimes\mathbf{P}_{\alpha'\beta'}(x)$ and the special form of the spectral decomposition of $\mathbf{P}_{\alpha\beta}(x)$. By the same algebraic manipulations, the explicit form of the left and right eigenvectors of $\mathbf{P}$ is found. The eigenvectors enter the coefficients $(\pi^*\mathbf{I}_1\mathbf{u}^{(r)})(\mathbf{v}^{(r)}\mathbf{I}_2\eta)$, which are provided in the main text.

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